§4.4 Renormalization away from the critical point and critical exponents

Up to now we have normalization conditions

$$\Gamma_R^{(2)}(0;g) = 0$$

$$\frac{\partial}{\partial k^2}\Gamma_R^{(2)}(k;g)\Big|_{k^2=\kappa^2} = 1 \qquad (1)$$

$$\Gamma_R^{(4)}(k_i;g)\Big|_{SP} = g$$

$\longrightarrow$ correspond to renormalized mass $m^2=0$

Recall that for the Ising model we saw

$$\underset{\substack{\uparrow\\ \text{susceptibility}}}{\chi} = G_0(k=0) = \mu^{-2} \quad \text{or} \quad \chi^{-1} = \mu^2$$

where

$$\mu^2 \equiv \frac{1}{\rho^2}\,\frac{T-T_0}{T_0}$$

is the square of the "free mass" and is a linear measure of the temperature.

$\longrightarrow$ free theory shows phase transition at $T \sim T_0$ with critical exponent $\gamma = 1$

or $\mu^2 \sim 0$

Turning on interactions with bare coupling λ changes this picture:

at one-loop: $\chi^{-1} = \mu^2 + \frac{\lambda}{2} \int dq \, \frac{1}{q^2+\mu^2}$

$\longrightarrow$ critical temperature is lowered:

$$\mu_c^2 = T_c - T_0 = -\frac{\lambda}{2} \int dq \, \frac{1}{q^2+\mu_c^2} \qquad (2)$$

RG flow gives scaling behaviour at $m^2(\mu^2 = \mu_c^2) = 0$ (compare eqs. (1))

$\longrightarrow$ to extract critical exponents, need to perturb away from this point!

introduce

$$\mu^2 \phi^2 = \mu_c^2 + (\mu^2 - \mu_c^2)\phi^2 = \mu_c^2 + \delta\mu^2 \phi^2 \qquad (3)$$

with $\delta\mu^2 \ll 1$

Then we can expand the bare vertex

$$\Gamma^{(N)}(k_i; \mu^2, \lambda, \Lambda) = \sum_{M=0}^{\infty} \frac{1}{M!} (\delta\mu^2)^M \, \Gamma^{(N,M)}(k_i; \mu_c^2, \lambda, \Lambda) \qquad (4)$$

What is $\Gamma^{(N,M)}$?

Recall definition of connected Green function:

$$G_c^{(N)}(x_1, \dots, x_N) \equiv \left. \frac{\delta^N W[J]}{\delta J(x_1) \cdots \delta J(x_N)} \right|_{J=0}$$

Now add to the Lagrangian the interaction

$$\mathcal{L}_{Int} = -\int \frac{t(y)}{2!} \phi^2(y)\, dy \qquad (5)$$

$\longrightarrow$ path integral over ϕ gives free energy

$$W[J,t] = -i \ln Z[J,t]$$

and we can set

$$G_c^{(N,L)}(x_1,\dots,x_N, y_1,\dots,y_L; t)$$

$$\equiv \left. \frac{\delta^{N+L} W[J,t]}{\delta J(x_1)\cdots\delta J(x_N)\,\delta t(y_1)\cdots\delta t(y_L)} \right|_{J=0}$$

from which we get

$$G_c^{(N,L+1)}(x_1,\dots,x_N, y_1,\dots,y_L; t)$$

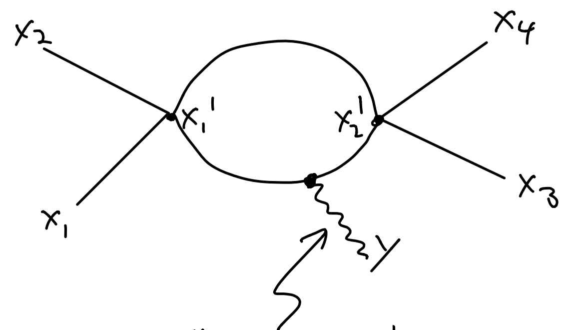
$$= \frac{\delta G_c^{(N,M)}(x_1,\dots,x_N,y_1,\dots,y_L;t)}{\delta t(y_{L+1})}$$

$$\longrightarrow G_c^{(N,M)}(x_i,y_i,t) = \sum_{K=0}^{\infty} \frac{1}{K!} \int dy_{L+1}\cdots dy_{L+K}\, t(y_{L+1})\cdots t(y_{L+K})$$
$$\times\, G_c^{(N,L+K)}(x_i, y_1,\dots,y_L, y_{L+1},\dots,y_{L+K})$$

where we have defined

$$G_c^{(N,L)}(x_i, y_i] \equiv \left. G_c^{(N,L)}(x_i, y_i; t)\right|_{t=0}$$

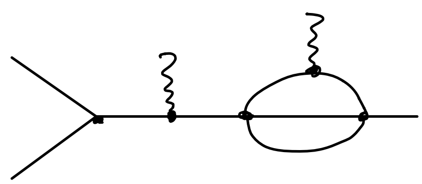
Pictorially, a graph of $G_c^{(4,1)}$ may look like:



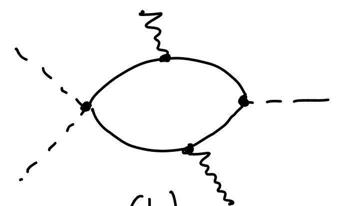
"composite operator" ϕ^2

Define 1PI part of $G_c^{(N,L)}(t)$ as $\Gamma^{(N,L)}(t)$

$\longrightarrow$ graphs of $G_c^{(3,2)}$ and $\Gamma^{(3,2)}$:



(a)



(b)

and we have:

$$\Gamma^{(N)}(x_1, \dots, x_N; t) = \sum_{L=0}^{\infty} \frac{1}{L!} \int dy_1 \cdots dy_L \, \Gamma^{(N,L)}(x_1, \dots, x_N, y_1, \dots, y_L) \times t(y_1) \cdots t(y_L)$$

and

$$\Gamma^{(N,L)} = \left. \frac{\delta^{N+L} \Gamma[\bar{\Phi}, t]}{\delta \bar{\Phi}(x_1) \cdots \delta \bar{\Phi}(x_N) \delta t(y_1) \cdots \delta t(y_L)} \right|_{J=t=0}$$

where $\quad \Gamma[\bar{\Phi}, t] = \sum \bar{\Phi}^i J_i - W[J, t]$

Applying to our current situation with

$$\mu^2\phi^2 = \mu_c^2\phi^2 + \delta\mu\phi^2$$

composite operator

we get

$$(5)\quad \Gamma^{(N)}(k_i;\mu^2,\lambda,\Lambda) = \sum_{M=0}^{\infty} \frac{1}{M!}(\delta\mu^2)^M \Gamma^{(N,M)}(k_i;q_i=0;\mu_c^2,\lambda,\Lambda)$$

For $\Lambda \to \infty$ there will be divergences

→ renormalize by writing

$$\Gamma_R^{(N,M)}(k_i\, q_i; g, \kappa) = Z_\phi^{N/2} Z_{\phi^2}^{M} \Gamma^{(N,M)}(k_i q_i; \mu_c^2, \lambda, \Lambda)$$

with $\delta\mu^2 = T - T_c = Z_{\phi^2}\,\delta t$

→ (5) becomes

$$\Gamma_R^{(N)}(k_1,\dots,k_N; t, u, \kappa)$$

$$= \lim_{q_i\to 0} \sum_{M=0}^{\infty} \frac{1}{M!} t^M \Gamma_R^{(N,M)}(k_i, q_i; u, \kappa)$$

Using

$$\left[\kappa\frac{\partial}{\partial\kappa} + \beta\frac{\partial}{\partial u} - \frac{1}{2}N\gamma_\phi + M\gamma_{\phi^2}\right]\Gamma_R^{(N,M)}(k_i, q_i; u, \kappa) = 0$$

we get

$$\left[\kappa\frac{\partial}{\partial\kappa} + \beta(u)\frac{\partial}{\partial u} - \frac{1}{2}N\gamma_\phi(u) + \gamma_{\phi^2}(u)\, t\frac{\partial}{\partial t}\right]\Gamma_R^{(N)}(k_i; t, u, \kappa) = 0 \quad (6)$$

At the stationary point $u = u^*$ with $\beta(u^*) = 0$ this eq. becomes

$$(7) \quad \left[\kappa \frac{\partial}{\partial \kappa} - \frac{1}{2} N \eta - \theta t \frac{\partial}{\partial t} \right] \Gamma_R^{(N)} (k_i; t, u^*, \kappa) = 0$$

where $\theta = -\gamma_{\phi^2}(u^*)$

One can show (without proof) that solutions of (7) are of the form

$$\Gamma_R^{(N)} (k_i; t, \kappa) = \kappa^{d + \frac{N}{2}(2-d)} (\kappa^{-2} t)^{\frac{d + N(2-d-\eta)/2}{\theta + 2}} F^{(N)}\left(\kappa^{-1} k_i \, (\kappa^{-2} t)^{-\frac{1}{\theta+2}} \right)$$

$\to$ homogeneous function of

$$\xi \sim t^{-1/(\theta+2)}$$

$\to$ critical exponents are extracted from

$$t^{-1/(\theta+2)} \sim |T_c - T|^{-\frac{1}{\theta+2}} = |T - T_c|^{-\nu}$$

$$\Rightarrow \nu^{-1} = 2 + \theta = 2 - \gamma_{\phi^2}^*$$

Other critical exponents can be obtained from ν and η:

- $\chi \sim C|T - T_c|^{-\gamma}$, $T > T_c$ and $\gamma = \nu(2-\eta)$
- specific heat

  $C \sim A|T - T_c|^{-\alpha}$, $T > T_c$ and $\nu d = 2 - \alpha$

§4.5 The Callan-Symanzik equations

Consider the normalization conditions

$$\Gamma_R^{(2)}(0; m^2, g) = m^2$$

$$\frac{\partial}{\partial k^2}\Gamma_R^{(2)}(0; m^2, g) = 1$$

$$\Gamma_R^{(4)}(k_i = 0; m^2, g) = g$$

$$\Gamma_R^{(2,1)}(k_i = 0, m^2, g) = 1$$

→ independent of k (set to 0 here), while depending on m^2

→ renormalization constants will depend on ratio m/Λ

$$\Gamma_R^{(N)}(k_i; m^2, g) = Z_\phi^{N/2}\Gamma^{(N)}(k_i; \mu^2, \lambda, \Lambda)$$

→ dimensional analysis implies

$$\mu^2 = m^2 \bar{\mu}^2(u, m/\Lambda),$$

$$\lambda = m^\varepsilon u_0(u, m/\Lambda),$$

$$Z_\phi = Z_\phi(u, m/\Lambda)$$

with $g = m^\varepsilon u$

→ $\Gamma_R^{(N)}$ is function of k_i, m^2 and u

satisfies diff. eq. :

$$\left[m\frac{\partial}{\partial m}+\beta(u)\frac{\partial}{\partial u}-\frac{N}{2}\gamma_\phi(u)\right]\Gamma_R^{(N)}(k_i;m^2,u)$$

$$=Z_\phi^{N/2}\, m\left(\frac{\partial\mu^2}{\partial m}\right)_{\lambda,\Lambda}\frac{\partial}{\partial\mu^2}\Gamma^{(N)}(k_i;\mu^2,\lambda,\Lambda)$$

with
$$\beta(u)=\left(m\frac{\partial u}{\partial m}\right)_{\lambda,\Lambda}=-\varepsilon\left(\frac{\partial\ln u_0}{\partial u}\right)^{-1}$$

$$\gamma_\phi(u)=m\left(\frac{\partial\ln Z_\phi}{\partial m}\right)_{\lambda,\Lambda}=\beta(u)\,\frac{\partial\ln Z_\phi}{\partial u}$$